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World-sheet corrections via D-instantons

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Abstract: We use a D-instanton or physical gauge approach to re-derive the heterotic string worldsheet instanton contribution to the superpotential in Calabi-Yau compactification. We derive an analogous formula for worldsheet instanton corrections to the moduli space metric in heterotic string or type-I compactification on a K3 surface. In addition, we give a global analysis of the phase of the worldsheet path integral of the heterotic string, showing precisely how the B-field must be interpreted.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

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Contents

1. Introduction

World-sheet instanton corrections to string compactifications have been first computed [1] from the usual standpoint of worldsheet conformal field theory. In this approach, the worldsheet is an abstract Riemann surface Σ; one integrates over complex structures on Σ and maps of Σ to spacetime, and divides by the symmetries of Σ.

The conformal field theory approach to the path integral has the great virtue that it makes it possible to quantize strings microscopically. For example, the image of Σ in spacetime might be a single point; but this causes no difficulty in evaluating the path integral.

In worldsheet instanton physics, at least if the instanton is a smooth submanifold C of a spacetime X, understanding what a collapsed string would do is not so important. Here one is evaluating the contribution to the path integral from *embeddings* of C in X. There is then another possible point of view about worldsheet instantons, which we might call the physical gauge approach. In this approach, one views the worldsheet instanton as a submanifold $C \subset X$, and integrates only over its physical or transverse oscillations. This avoids the redundancy that is present in the conformal field theory approach.

The physical gauge approach to worldsheet instantons was formulated in [2] in the process of developing a unified approach to string and brane instanton corrections. This required using a physical gauge approach because for the p-branes of $p > 1$ there is no (known) analog of the conformal field theory description. The physical gauge approach was used in [2] to compute instanton corrections to moduli space geometry in type-II compactification on a Calabi-Yau threefold. These corrections arise in a similar fashion from both strings and p-branes of $p > 1$. The string contributions are analogous to worldsheet instanton corrections to the heterotic string superpotential [1], and were originally discussed in the conformal field theory approach in [3].

The purpose of the present paper is to reconsider the heterotic string worldsheet corrections in a physical gauge approach. A natural and equivalent setting for the discussion (at least in the case of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string) is to consider D-instanton contributions to type-I compactification. In section 2, we consider heterotic or type-I string compactification to four dimensions on a Calabi-Yau threefold, and analyze the D-instanton contributions to the superpotential. We get results equivalent to those of [1], but some properties are more obvious. In section 3, we consider heterotic or Type I compactifications with eight unbroken supercharges, for example, compactification to six dimensions on a K3 manifold or compactification to four dimensions on $\mathrm{K3} \times \mathbb{T}^2$. In this case (roughly as in type-II compactification on a Calabi-Yau threefold [3, 2], which also leaves eight unbroken supercharges), the instantons do not generate a superpotential, but rather correct the metric on the hypermultiplet moduli space.

In heterotic string compactification on K3, the dilaton is in a tensor multiplet, and hence the hypermultiplet metric is independent of the string coupling constant. The hypermultiplet metric can in principle, therefore, be computed exactly from heterotic string worldsheet conformal field theory ($(0, 4)$ conformal field theory, to be more exact). It differs, however, from the metric on hypermultiplet moduli space that would be computed in classical field theory; the differences are very likely determined precisely by the worldsheet instanton contributions that we will discuss. The wording of the last sentence reflects the fact that since the Einstein equations obeyed by a quaternionic metric are nonlinear, the exact quaternionic metric may somehow involve nonlinear combinations of instanton contributions.

The physical gauge approach to membrane contributions to the superpotential in compactification on a manifold of G_2 holonomy has been recently developed in [4]. The validity of the physical gauge approach is discussed in section 3 of that paper.

2. D-instantons in Calabi-Yau threefolds

2.1 Evaluation of the superpotential

We consider the heterotic or type-I superstring on $\mathbb{R}^4 \times X$, where X is a Calabi-Yau threefold. X is endowed with a Calabi-Yau metric (or its generalization in conformal field theory) and a suitable holomorphic $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ gauge bundle. As anticipated in the introduction, we will analyze the instanton contributions to the superpotential mainly from the D-instanton point of view, which means that we mainly consider the type-I or heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. The answer for $E_8 \times E_8$ is, however, also determined by the resulting formula, as we briefly explain below.

A note on vector structure. We can assume that the $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle has vector structure[1] — in other words, that it can be derived from an $SO(32)$ bundle V — at least in a neighborhood of the instanton. Indeed, a D-string cannot be wrapped on an oriented two-dimensional surface C unless the gauge bundle, restricted to C, has vector structure. This restriction has a natural interpretation in K-theory using ingredients described in [6, section 5.3]; a D-instanton wrapped on C represents a class in $\mathrm{KO}(\mathbb{R}^4 \times X)$, but if the obstruction $\widetilde{w}_2$ to vector structure is non-zero, then the allowed D-branes take values in a twisted KO-group $\mathrm{KO}_{\widetilde{w}_2}(\mathbb{R}^4 \times X)$. Hence a D-instanton can be wrapped on C only if KO and $\mathrm{KO}_{\widetilde{w}_2}$ coincide when restricted to C, that is, only if $\widetilde{w}_2$ vanishes when restricted to C.

A more down-to-earth explanation of the requirement that the bundle should have vector structure comes by considering the fermionic description of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string (or the D-string equivalent, which we will use below). The left-moving fermions are sections of $S_- \otimes V$, where S_- is the negative chirality spin bundle of C. Hence V, restricted to C, must exist (as a bundle in the vector representation), as claimed. If the restriction of V to C does not exist, then a path integral with a heterotic string worldsheet wrapped on C makes sense only if one inserts an odd number of "twist" fields that twist the left-moving fermions and transform in the spinor representation of Spin(32). These fields are vertex operators of massive particles, and contributions to the superpotential containing them are inessential.[2] From the type-I point of view, this means that if $\widetilde{w}_2$ is nonzero when restricted to C, then a D-instanton wrapped on C can be considered if and only if there terminate on C an odd number of worldlines of nonsupersymmetric D-particles [7] transforming as spinors of Spin(32). Such a configuration is not supersymmetric and will not contribute to a superpotential.

The gauge bundle V must, in addition, have $w_2(V) = 0$, because of the existence of the massive particles just mentioned which transform as spinors of Spin(32).

Computation of the superpotential. In computing the instanton contribution to the superpotential, only holomorphic genus zero instantons are relevant, for familiar reasons of holomorphy [1]. We will in this paper only consider the case of an isolated instanton, though it is perhaps also important to consider the general case. (We assume actually that the instanton is isolated in a very strong sense: no bosonic zero modes except those that follow from translation symmetries of $\mathbb{R}^4$.) Moreover, we will consider only the case of a smooth instanton. Thus, our instanton will be a smooth isolated genus zero holomorphic curve $C \subset X$.

[1] See [5, section 4] for background about this concept.

[2] Fields that are massive for all values of the moduli can be integrated out of the superpotential by a holomorphic change of variables. Let Φ be such a massive field and Ψ_i an arbitrary collection of possibly light fields. In any superpotential $\frac{1}{2}M\Phi^2 + \Phi\Sigma(\Psi_i)$, with Σ a holomorphic function, Φ can be decoupled by $\Phi \to \Phi - \Sigma(\Psi_i)/M$.

The instanton has certain zero modes and collective coordinates that are easily described. Though C is isolated in X, it can be translated in $\mathbb{R}^4$, leading to four bosonic collective coordinates x^i. Also, while heterotic string compactification on X preserves four supercharges, two of each chirality, the instanton preserves the two of one chirality and violates the others. The two supersymmetries that are broken by the instanton lead to two fermion zero modes and collective coordinates θ^α. The term L_C in the effective action induced by an instanton C will hence be

$$L_C = \int d^4x d^2\theta \, \mathcal{W}_C \,, \tag{2.1}$$

where we have made explicit the integral $d^4xd^2\theta$ over the collective coordinates, and $\mathcal{W}_C$ is calculated by performing the world-sheet path integral with the bosonic and fermionic zero modes suppressed in the worldsheet path integral. The contribution W_C of C to the superpotential is obtained from $\mathcal{W}_C$ by setting all derivatives and fermions to zero.

The D-instanton path integral in the one-loop approximation takes the general form

$$\exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}'(\mathcal{D}_F)}{\sqrt{\det{}' \, \mathcal{D}_B}} \,. \tag{2.2}$$

Here, $A(C)$ is the area of the surface C using the heterotic string Kähler metric on X and α' is the heterotic string parameter. B is the B-field; in the type-I description, it is a Ramond-Ramond field, while in the heterotic string, it arises in the Neveu-Schwarz sector. The exponential terms in (2.2) come from the classical instanton action, while the other factors represent the one-loop integral over quantum fluctuations around the classical instanton solution. $\mathcal{D}_F$ and $\mathcal{D}_B$ are the kinetic operators for the bosonic and fermionic fluctuations, respectively. The path integrals give the Pfaffian for fermions and the square root of the determinant for bosons; the "prime" in Pfaff$'$ and det$'$ means that the zero modes associated with collective coordinates are to be omitted. All determinants and Pfaffians are computed using the metric that C obtains as a submanifold of X, so there is no issue of a conformal anomaly (this contrasts with the conformal field theory formulation, in which an abstract metric on C is introduced, and additional ghost determinants cancel the conformal anomaly). Arguments of holomorphy (which are most familiar and perhaps most transparent in the heterotic string description [1]) show that the superpotential receives no contributions from higher order corrections to the path integral, so that for purposes of computing it, the integrals over the small fluctuations in fact reduce to determinants. The multi-loop contributions to the worldsheet path integral give a plethora of higher-derivative interactions, but do not contribute to the superpotential.

Let S_+ and S_- be the right- and left-handed spin bundles of C. We pick the complex structure so that the kinetic operator for a left-moving fermion (a section of S_-) is a $\overline{\partial}$ operator, while that for a right-moving fermion is a ∂ operator. Let N

denote the normal bundle to C in $\mathbb{R}^4 \times X$, understood as a rank eight real bundle, and let $S_+(N)$ denote the positive chirality spinors of N. Right-moving fermions are sections of $S_+ \otimes S_+(N)$, while left-moving fermions are sections of $S_- \otimes V$ (with V being as before the SO(32) gauge bundle).

The eight real bosons representing transverse oscillations in the position of $C \subset \mathbb{R}^4 \times X$ can, in a fairly natural way, be grouped as four complex bosons. The normal bundle to C in X has a natural complex structure; in fact, it must be isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in order for C to be isolated.[3] If one picks a complex structure on $\mathbb{R}^4$, then the trivial rank four bundle representing the $\mathbb{R}^4$ part of the normal bundle can similarly be written as $\mathcal{O} \oplus \mathcal{O}$. If we reinterpret the real operator $\mathcal{D}_B$ acting on eight real bosons as a complex operator $\mathcal{D}'_B$ acting on four complex bosons, then we can write $\sqrt{\det \mathcal{D}_B} = \det \mathcal{D}'_B$.

In (2.2), because of the unbroken supersymmetry of the instanton field, the non-zero modes of the right-moving fermions cancel the right-moving modes of the bosons. The Dirac operator for left-moving fermions is the $\overline{\partial}$ operator on $S_- \otimes V$, which equals $\mathcal{O}(-1) \otimes V$ (since $S_- = \mathcal{O}(-1)$ as a holomorphic bundle). We abbreviate this as $V(-1)$. The left-moving part of D'_B is the $\overline{\partial}$ operator on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}$. So (2.2) becomes

$$W_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}(\overline{\partial}_{V(-1)})}{(\det\, \overline{\partial}_{\mathcal{O}(-1)})^2 (\det'\, \overline{\partial}_{\mathcal{O}})^2} \tag{2.3}$$

and this, in fact, is our formula for the superpotential. Note that by approximating the worldsheet path integral with the one-loop determinants, we have dropped the higher derivative interactions and reduced the more general action $\mathcal{W}_C$ to the superpotential W_C. The exponent in the first factor in W_C is roughly $\exp(-\mathcal{A}_C)$, where $\mathcal{A}_C$ is the superfield

$$\mathcal{A}_C = \frac{A(C)}{2\pi\alpha'} - i\int_C B\,. \tag{2.4}$$

We give a more careful discussion of the phase factor in (2.3), or in other words the additive constant in $\mathcal{A}_C$, in section 2.2.

The most striking difference between this derivation and the analogous derivation based on conformal field theory is perhaps that in conformal field theory, it is most natural to compute the third derivative of the superpotential, while here we obtain directly a formula for W_C. In practice, this does not make much difference, since one can take the third derivative with respect to $\mathcal{A}$, on which W_C has a known exponential dependence, and thereby compute a third derivative of W_C without losing any information. But it is clearly desireable to be able to compute W_C directly.

[3] The notation is standard: $\mathcal{O}(n)$ is a holomorphic line bundle whose sections are functions homogeneous of degree n in the homogeneous coordinates of $C \cong \mathbb{CP}^1$. In particular, $\mathcal{O}(0) = \mathcal{O}$ is a trivial complex line bundle.

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Condition for vanishing. The most striking property of the formula for W_C, already known [8] from the conformal field theory point of view, is the following. In the denominator in (2.3), we still have a $\det'$ for bosons to indicate that the constant zero mode of $\overline{\partial}_{\mathcal{O}}$, which is associated with the translational collective coordinates, should be removed. However, the fermion collective coordinates are zero modes of right-moving fermions, and are absent in the formula (2.3), which contains determinants of left-movers only. Hence the Pfaffian in the numerator in (2.3) is a full Pfaffian. The contribution W_C of the instanton C to the superpotential therefore vanishes if and only if the Pfaffian of $\overline{\partial}_{V(-1)}$ vanishes, or in other words if and only if this operator has a nonempty kernel.

Any SO(32) bundle V over a genus zero curve C is of the form

$$V = \bigoplus_{i=1}^{16} \left(\mathcal{O}(m_i) \oplus \mathcal{O}(-m_i)\right), \tag{2.5}$$

with nonnegative integers m_i that are uniquely determined up to permutation. So

$$V(-1) = \bigoplus_{i=1}^{16} \left(\mathcal{O}(m_i - 1) \oplus \mathcal{O}(-m_i - 1)\right). \tag{2.6}$$

Since $\overline{\partial}_{\mathcal{O}(s)}$ has a kernel of dimension $s+1$ for all $s \geq 0$, and otherwise zero, the dimension of the kernel of $\overline{\partial}_{V(-1)}$ is

$$\Delta = \sum_{i=1}^{16} m_i \,. \tag{2.7}$$

Δ vanishes if and only if the m_i are all zero, that is if $V|_C$ (the restriction of V to C) is trivial. In any event, Δ is always even; this follows from the requirement $w_2(V) = 0$, since in general

$$(w_2(V), C) = \sum_{i=1}^{16} m_i \bmod 2 \,. \tag{2.8}$$

Hence W_C vanishes if and only if $V|_C$ is nontrivial. The condition for W_C to be stationary with respect to variations in the gauge field is stronger. A first order perturbation of the fermion kinetic operator can lift a pair of fermion zero modes, so to ensure vanishing of W_C and all its first derivatives, one needs $\Delta > 2$.

The case $E_8 \times E_8$. What if we consider the heterotic string with gauge group $E_8 \times E_8$ instead of $\mathrm{Spin}(32)/\mathbb{Z}_2$? The fermionic construction of the heterotic string makes it obvious that the formula (2.3) still holds if the structure group of the bundle restricts to SO(16) × SO(16), which can be naturally regarded[4] as a subgroup of

[4] We are being slightly imprecise here and not distinguishing the various global forms SO(16), Spin(16), and $\mathrm{Spin}(16)/\mathbb{Z}_2$. The justification for this is that, as explained earlier, when restricted to C the obstructions w_2 and $\widetilde{w}_2$ vanish.

either $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. Moreover, holomorphy says that the superpotential is invariant under complexified $E_8 \times E_8$ gauge transformations. On a genus zero curve, the classification of holomorphic bundles says that by a complexified gauge transformation, the structure group of any G-bundle (for any semisimple gauge group G) can be reduced to a maximal torus; for $E_8 \times E_8$, a maximal torus coincides with that of $\mathrm{SO}(16) \times \mathrm{SO}(16)$. So the $\mathrm{Spin}(32)/\mathbb{Z}_2$ result (2.3) together with gauge invariance and holomorphy uniquely determines the result also for $E_8 \times E_8$.

It seems extremely difficult to give an elegant formula for the $E_8 \times E_8$ analog of the Pfaffian, but one can give a theoretical explanation of what the answer means. The Pfaffian $\mathrm{Pfaff}(\overline{\partial}_{V(-1)})$ is the partition function of $\mathrm{Spin}(32)/\mathbb{Z}_2$ current algebra at level one, coupled to a background gauge field, and the analog for $E_8 \times E_8$ is simply the partition function of $E_8 \times E_8$ current algebra at level one, coupled to a background gauge field.

Multiple covers. One important area where the conformal field theory and D-instanton derivations look quite different, at least at first sight, is in treating multiple covers of C. To study k-fold instanton wrapping on C, in conformal field theory, one must integrate over the moduli space of k-fold holomorphic covers of C. An elegant result has been obtained, at least for $(2,2)$ models [9, 10]. In the D-instanton approach, to study a k-fold cover, one must endow the D-instanton with Chan-Paton factors of the gauge group $\mathrm{SO}(k)$. As a result, one must study a certain supersymmetric $\mathrm{SO}(k)$ gauge theory on C. To get the superpotential, however, this must be studied only in the limit of weak coupling. The analysis might be tractable, though it is beyond the scope of the present paper.

We can also reconsider, for multicovers, the condition that the restriction of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle to C must admit vector structure if C is to contribute to the superpotential. For a k-fold cover, the left-moving fermions on the D-instanton world-volume are sections of $S_- \otimes V \otimes W$ where V is the $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle in the vector representation, and W is the $\mathrm{SO}(k)$ Chan-Paton bundle, also in the vector representation. So the condition is not that V or W should exist separately in the vector representations, but that the tensor product $V \otimes W$ should exist in the tensor product $\mathbf{32} \otimes \mathbf{k}$ of those representations. If the $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle restricted to C does not admit vector structure, then likewise the $\mathrm{SO}(k)$ Chan-Paton bundle on C must not admit vector structure. This implies that k must be even.[5] In this case, the Chan-Paton bundle cannot be flat, and the classical action of the instanton has an additional term from its curvature. It appears that such a configuration is not supersymmetric and does not contribute to the superpotential.

[5] The ability to construct an $\mathrm{SO}(k)$ bundle without vector structure depends on having an element -1 of the center of $\mathrm{SO}(k)$ that acts nontrivially in the vector representation. Such an element exists only for even k; for odd k, the element -1 of $O(k)$ does not lie in $\mathrm{SO}(k)$.

From the conformal field theory point of view, the role of even k arises because if $\phi : \Sigma \to C$ is a degree k map, then $\phi^*(\widetilde{w}_2)$ is always zero for even k (even if $\widetilde{w}_2$ is not), so for even k, $\phi^*(V)$ always makes sense in the vector representation. From this point of view, it appears that k-fold wrappings for even k might contribute to the superpotential.

2.2 The phase of the superpotential

This completes what we will say about the heterotic string or D-instanton superpotential, except for a technical analysis of the phase factor in (2.3). (This discussion is not needed as background for section 3.) In this discussion, we consider the heterotic string or D-instanton in an arbitrary spacetime Y, with gauge bundle V, subject only to the usual anomaly cancellation conditions, and an arbitrary (closed and oriented) string worldsheet C. (To make contact with the discussion of the superpotential, the analysis can be specialized to $Y = \mathbb{R}^4 \times X$ with X a Calabi-Yau manifold and C a holomorphic curve in X.) Most of the discussion below is a summary of standard material concerning heterotic string worldsheet anomalies. However, a complete definition of the overall phases of the heterotic string path integral in the different topological sectors has apparently never been given in the literature. For this, we will need a theorem of Dai and Freed [11] which generalizes the formulas for computing global anomalies.

Naively speaking, the B-field is a two-form, and the phase factor

$$\exp\left(i\int_C B\right) \tag{2.9}$$

is a complex number. However, life is really much more subtle. The field strength, naively $H = dB$, of the B-field, does not obey the expected Bianchi identity $dH = 0$, but rather obeys the equation

$$dH = \frac{1}{4\pi}\left(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F\right). \tag{2.10}$$

Here R is the Riemann tensor of spacetime, and F is the curvature of the SO(32) connection on V. (2.10) is associated with heterotic string anomaly cancellation in the following sense. Let $\lambda(Y)$ and $\lambda(V)$ be the characteristic classes of the tangent bundle of Y and of V which at the level of differential forms are represented by $(1/8\pi^2)\operatorname{tr} R \wedge R$ and $(1/8\pi^2)\operatorname{tr} F \wedge F$. A three-form H obeying (2.10) exists if and only if $\lambda(Y) = \lambda(V)$ mod torsion; existence of such an H is required (as we will review below) for cancellation of perturbative heterotic string worldsheet anomalies. The stronger condition that

$$\lambda(Y) = \lambda(V) \in H^4(Y;\mathbb{Z}) \tag{2.11}$$

gives cancellation of global worldsheet anomalies.

Obviously, given (2.10), B is not a potential for the gauge-invariant three-form H in the naive sense $H = dB$, for this would imply $dH = 0$. The familiar formula for H is

$$H = dB + \omega_{\rm grav} - \omega_{\rm gauge}\,, \tag{2.12}$$

where $\omega_{\rm grav}$ and $\omega_{\rm gauge}$ are suitably normalized gravitational and gauge Chern-Simons three-forms. Consequences of this formula for B will be discussed presently, but first note the following fact, which we will need later. For any closed three-cycle W in spacetime,

$$\int_W H = \int_W (\omega_{\rm grav} - \omega_{\rm gauge}) \bmod 2\pi\,. \tag{2.13}$$

We are here using the fact that though the three-forms $\omega_{\rm grav}$ and $\omega_{\rm gauge}$ are not gauge-invariant, their periods are gauge-invariant modulo 2π.

So what kind of object is B? Under local Lorentz or gauge transformations with infinitesimal parameters θ and ϵ, one has the familiar formulas $\omega_{\rm grav} \to \omega_{\rm grav} + d({\rm tr}\,\theta R)$ and $\omega_{\rm gauge} \to \omega_{\rm gauge} + d({\rm tr}\,\epsilon F)$, so for gauge invariance of H, B transforms in the familiar fashion

$$B \to B - {\rm tr}\,\theta R + {\rm tr}\,\epsilon F\,. \tag{2.14}$$

In particular, B, and likewise the phase factor in (2.9), is not gauge-invariant.

We will see that it is a long story to explain what a B-field actually is. One simple thing that we can say right away is the following. Let us agree that by an ordinary two-form field we mean a field that is locally represented by a two-form B', with field strength $H' = dB'$ and standard Bianchi identity $dH' = 0$, and subject to the usual integrality conditions on the periods.[6] Then B is not an ordinary two-form field, but the space of B-fields is a "torsor" for the group of ordinary two-form fields. This is a fancy way to say that to B we can add an ordinary two-form field B', and that given one B field (with given Y and V), any other B field is of the form $B + B'$ for some unique B'.

This statement alone does not determine what the phase of the path integral is supposed to be. In fact, we will only make sense of the B-field phase factor (2.9) in conjunction with another factor in the worldsheet path integral. The other relevant factor is of course the fermion Pfaffian. We must understand the product

$$\text{Pfaff}(\mathcal{D}_F) \cdot \exp\left(i \int_C B\right). \tag{2.15}$$

(We here omit the bosonic determinant, which is positive and contributes no interesting phase. The operator $\mathcal{D}_F$ includes both left and right-moving fermions.)

We therefore must discuss the phase of $\text{Pfaff}(\mathcal{D}_F)$. The Pfaffian of $\mathcal{D}_F$ is not, in general, well-defined as a complex number. It takes values in a complex line that

[6] Mathematically, such a B' is sometimes called a connection on a gerbe. The B-fields of type-II superstring theory are such fields.

varies, as C varies, to give a Pfaffian line bundle over the space of C's. We let $\mathcal{L}_{[C]}$ denote the Pfaffian line bundle for worldsheets in the homotopy class of C.[7] For Dirac operators in general, a Pfaffian line bundle such as $\mathcal{L}_{[C]}$ might be topologically nontrivial. For the heterotic or D-string, the condition (2.11) ensures that $\mathcal{L}_{[C]}$ is topologically trivial.

However, it does not have a canonical trivialization, and hence $\mathrm{Pfaff}(\mathcal{D}_F)$ is not defined as a complex number. There is always a natural absolute value $|\,\mathrm{Pfaff}(\mathcal{D}_F)|$ (defined, for example, by zeta function regularization), so the problem is only with the phase $f(C) = \mathrm{Pfaff}(\mathcal{D}_F)/|\,\mathrm{Pfaff}(\mathcal{D}_F)|$. While lacking a canonical trivialization, $\mathcal{L}_C$ does have a canonical unitary connection θ [12]. This means that if we are given two world-sheets C_1 and C_2 in the same homotopy class $[C]$, and a path P between them, then we can define the phase of $\mathrm{Pfaff}(\mathcal{D}_F)$ at C_2 in terms of what it is at C_1. Concretely, such a path P is a three-manifold $U \subset Y$ of topology $C \times I$, with $I = [0, 1]$ being a unit interval on the x-axis, such that U coincides with C_2 at $x = 1$ and with C_1 at $x = 0$. (More generally, we could define a path by a map $\phi : C \times I \to Y$ with $\phi(C \times 1) = C_2$ and $\phi(C \times 0) = C_1$.) Given such a path, we set

$$f_P(C_2) = \exp\left(i \int_P \theta\right) f(C_1)\,. \tag{2.16}$$

As we have suggested in the notation, $f_P(C_2)$ depends on the path P, because the connection θ is not flat. If P is deformed to another path P' keeping the endpoints fixed, then the Quillen-Bismut-Freed formula for the curvature of θ asserts that

$$\exp\left(i \int_{P'} \theta\right) = \exp\left(-i \int_K \frac{\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F}{4\pi}\right) \exp\left(i \int_P \theta\right), \tag{2.17}$$

where K is the four-manifold swept out by U in varying the path from P to P'.

At this point, the shifted Bianchi identity (2.10) saves the day. We modify the connection θ by adding an extra term involving the integral of H, replacing the phase factor in (2.16) by the product

$$\exp\left(i \int_P \theta\right) \exp\left(i \int_U H\right). \tag{2.18}$$

By virtue of (2.10) and (2.17), this product is invariant under continuous variations of P; in other words, the modified connection on $\mathcal{L}_{[C]}$ is flat. This statement is, in fact, equivalent to cancellation of heterotic string perturbative anomalies.

Though we have formulated the discussion in seemingly abstract terms involving connections on determinant line bundles, we have by now implicitly arrived at a partial explanation of the meaning of the phase factor in (2.9). The variation of this

[7] The square of the Pfaffian line bundle is a more familiar determinant line bundle. We get a Pfaffian line bundle because the heterotic string worldsheet fermions are Majorana-Weyl fermions.

factor, when C varies along a path P to sweep out a three-manifold U, should be understood as

$$\exp\left(i\int_U H\right), \tag{2.19}$$

and thus we know what is meant by the variation along a path of $\exp(i\int_C B)$, though we cannot make sense of $\exp(i\int_C B)$ itself. Thus, if we set $F(C)=f(C)\exp(i\int_C B)$, the phase factor (2.18) can be understood physically as describing parallel transport of $F(C)$ along the path P:

$$F(C_2)=\exp\left(i\int_P \theta\right)\exp\left(i\int_U H\right)F(C_1)\,. \tag{2.20}$$

Cancellation of *global* worldsheet anomalies is the assertion that the modified connection on $\mathcal{L}_{[C]}$ also has trivial holonomies globally, so that the above formula for $F(C_2)$ in terms of $F(C_1)$ is invariant even under discontinuous changes in P. We will briefly recall the proof [13, 14]. (The following summary omits some important steps. The goal is just to write down a couple of formulas that will be handy later.) If P is a closed path, then one can glue together the ends of U (which are copies of the same surface C) to make a closed three-manifold $W\subset Y$. The holonomy around W of the connection θ is, by the global anomaly formula,

$$\exp\left(\frac{i\pi\eta(W)}{2}\right), \tag{2.21}$$

where $\eta(W)$ is the eta-invariant of a suitable Dirac operator on W. Including also the contribution of H, the holonomy of the modified connection will vanish if

$$\exp\left(\frac{i\pi\eta(W)}{2}\right)\cdot\exp\left(i\int_W H\right)=1\,. \tag{2.22}$$

That this holds, for all W, is proved by using the Atiyah-Patodi-Singer theorem to evaluate the η-invariants, and using (2.13) to evaluate $\int_W H$.

Given that the modified connection is completely trivial, we get a complete definition of the phase factor in the path integral

$$\mathrm{Pfaff}(\mathcal{D}_F)\exp\left(i\int_C B\right) \tag{2.23}$$

for every worldsheet C in a given homotopy class $[C]$, in terms of a choice of phase $F(C_1)$ at an arbitrary base-point C_1 in the homotopy class. But what is the phase $F(C_1)$?

If C_1 is nontrivial in $H_2(Y;\mathbb{Z})$, then this question does not have an answer that depends only on the gauge-invariant field H (plus the metric and connection on Y and V), because the answer depends on the B-field, which is not fully specified even when H is known. It is elusive to explain what a B-field is, but as we remarked above, B-fields can be transformed by $B \to B + B'$, with B' an "ordinary two-form field". If H is given, we are limited to $B \to B + B'$ with B' *flat*, so B' defines an element of $H^2(Y; \mathrm{U}(1))$. Under $B \to B + B'$, $F(C_1)$ is multiplied by $\exp(i \int_{C_1} B')$, and if C_1 is nontrivial in $H_2(Y; \mathbb{Z})$, there is a flat B' field for which this factor is not 1.

More generally, suppose that $C_1, C_2, \ldots, C_s$ are a set of worldsheets that are linearly independent (they obey no linear relations with integer coefficients) in $H_2(Y; \mathbb{Z})$. Then, as the B-field is varied keeping H fixed, the phases $F(C_1)$, $F(C_2), \ldots, F(C_s)$ vary completely independently. This is because the phases $\exp\left(i \int_{C_i} B'\right)$, for flat B', are completely independent if the C_i are linearly independent in homology.

Suppose, on the other hand, that $C_1, \ldots, C_s$ do obey a linear relation. There is no essential loss of generality in assuming that this linear relation is

$$C_1 + C_2 + \cdots + C_s = 0\,. \tag{2.24}$$

(If some coefficients are negative, we reverse the orientations of the relevant C's; if some coefficients are bigger than 1, we increase s to reduce to the case that all coefficients are 1.) Such a relation means that there is a three-manifold $U \subset Y$ whose boundary is the union of the C_i (or more generally a three-manifold U with a map $\phi : U \to Y$ such that the boundary of U is mapped diffeomorphically to the union of the C_i). In this situation, we can give a relation, which depends only on the gauge-invariant H-field and not on the mysterious B-field, for the product $\prod_{i=1}^{s} F(C_i)$.

First of all, though the factors $\exp\left(i \int_{C_i} B\right)$ are mysterious individually, for their product we can write an obvious classical formula that depends only on H and U:

$$\prod_{i=1}^{s} \exp\left(i \int_{C_i} B\right) = \exp\left(i \int_U H\right). \tag{2.25}$$

This expression depends on U, though this is not shown in the notation on the left hand side.

More subtle is the product of the Pfaffians. We recall that each fermion path integral $\mathrm{Pfaff}(\mathcal{D}_F(C_i))$ takes values in a complex line $\mathcal{L}_{C_i}$. However, according to a theorem of Dai and Freed [11], for every choice of a three-manifold U whose boundary is the union of the C_i (together with an extension of all of the bundles over U), there is a canonical trivialization of the product $\otimes_i \mathcal{L}_{C_i}$. This trivialization is obtained by suitably interpreting the quantity $\exp(i\pi\eta(U)/2)$, where $\eta(U)$ is an eta-invariant of a Dirac operator on U defined using global (Atiyah-Patodi-Singer) boundary conditions on the C_i. We write the trivialization as $T_U : \otimes_i \mathcal{L}_{C_i} \to \mathbb{C}$. Via this trivialization, the product $f(C_1)f(C_2)\ldots f(C_s)$ is mapped to a well-defined (but U-dependent) complex number $T_U(f(C_1) \otimes f(C_2) \otimes \cdots \otimes f(C_s))$. We now set

$$\prod_{i=1}^{s} \left(\mathrm{Pfaff}(\mathcal{D}_F(C_i)) \exp\left(i \int_{C_i} B \right) \right) = T_U(f(C_1) \otimes f(C_2) \otimes \cdots \otimes f(C_s)) \exp\left(i \int_U H \right). \tag{2.26}$$

The point is that, in fact, the right hand side is independent of the choice of U. For continuous variations of U, this follows from the variational formula that expresses a change in η (and hence in T_U) in terms of $\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F$, together with the Bianchi identity (2.10) which implies a similar formula for the variation of $\int_U H$. More generally, if U is replaced with another three-manifold U' with the same boundary (U' may or may not be in the same homotopy class as U), we let W be the three-manifold without boundary obtained by gluing together U and U' along their boundaries with opposite orientation. Then to prove that the right hand side of (2.26) is unchanged if U is replaced by U', we need the formula

$$T_{U'} = T_U \exp\left(i \int_W H \right). \tag{2.27}$$

This can proved as follows. One has $T_U = \exp(i\pi\eta(U)/2)$, $T_{U'} = \exp(i\pi\eta(U')/2)$ (where $\eta(U)$ and $\eta(U')$ are suitably interpreted η-invariants on the manifolds with boundary U and U'). The gluing formula for the eta-invariant [11] gives

$$\exp\left(\frac{i\pi\eta(W)}{2}\right) = \exp\left(\frac{i\pi\eta(U)}{2}\right) \exp\left(\frac{-i\pi\eta(U')}{2}\right). \tag{2.28}$$

(The minus sign in the last factor enters because, in gluing U and U' to make W, one reverses the orientation on U'.) Using this, (2.27) is equivalent to (2.22). It is no coincidence that the verification of well-definedness of (2.26) is so closely related to the proof of absence of global anomalies. The Dai-Freed theorem is a generalization of the global anomaly formula (and essentially reduces to it when U is constructed from a path from C_1 to C_2) and enables us not just to prove that the heterotic string path integral is well-defined (as we can learn from the global anomaly formula) but to relate the path integrals in different topological sectors.

The nature of the B-field. At last, I can state the best description I know for what a B-field is for the heterotic string. A B-field is a choice of phases $F(C_i)$ for the heterotic string (or D-string) world-sheet path integral for all closed surfaces C_i, obeying (2.26) whenever $C_1 + \cdots + C_s$ is a boundary. As justification for this notion of a B-field, I note that with this definition, the heterotic string world-sheet path integral is manifestly well-defined for every choice of B-field. Moreover, it makes sense to transform $B \to B + B'$ for any ordinary two-form field B' (flat or not). This operation transforms $F(C_i) \to F(C_i) \exp\left(i \int_{C_i} B' \right)$, which together with $H \to H + dB'$ clearly preserves (2.26). Conversely, with this definition, any two B-fields are related by $B \to B + B'$ for a unique B'. (If (2.26) is obeyed with either of two B-fields B_1 and B_2, then the difference $B_1 - B_2$ obeys a relation similar to (2.26) but with the Pfaffians canceled out; this relation is equivalent to the defining property of an ordinary two-form field B'.) These properties together fully characterize what we want a B-field to be.

I believe that with this notion of a B-field, all other formulas in which the B-field appears for the heterotic or type-I string (like the Green-Schwarz anomaly-canceling terms) make sense.

3. Moduli space metric for $(0,4)$ models

We will now consider, with the methods of section 2.1, the heterotic (or type-I) string on $\mathbb{R}^6 \times Y$, with Y a K3 manifold. Other compactifications with eight unbroken supersymmetries, such as K3$\times\mathbb{T}^2$ compactification to four dimensions, can be treated similarly.

The parameters labeling the hyper-Kähler metric and B-field on Y, as well as the moduli of the gauge bundle, all transform in hypermultiplets. The metric on hypermultiplet moduli is independent of the heterotic string coupling constant, so it can be computed in the weak coupling limit, that is, from conformal field theory. Worldsheet instantons, or D-instantons, that correct this metric are therefore of genus zero. For the same reasons as in section 2, in the case of the Spin(32)/$\mathbb{Z}_2$ heterotic or type-I string, contributions come only from instantons C such that the gauge bundle V has vector structure when restricted to C.

A formula for the correction to the metric can be obtained along the lines of the analysis in section 2. Let $C \subset Y$ be a genus zero surface that is invariant under half of the supersymmetries. This is so if, and only if, C is holomorphic with respect to one of the complex structures on Y. If so, the D-instanton (or elementary heterotic string) wrapped on C has four fermionic zero modes, coming from the broken supersymmetries, as well as six bosonic zero modes representing translations in $\mathbb{R}^6$. The effective action L_C due to the instanton takes the general form

$$L_C = \int d^6x\, d^4\theta\, \mathcal{U}_C\,, \tag{3.1}$$

where $\mathcal{U}_C$ is computed from a worldsheet path integral with the zero modes suppressed. If $\mathcal{U}_C$ has a term U_C with no fermions or derivatives, then the integral $\int d^4\theta\, U_C$ will generate (among other things) terms $f_{ij}(\Phi)d\Phi^i d\Phi^j$, with Φ^i the bosonic part of the hypermultiplets. Such terms are the desired corrections to the hypermultiplet moduli space metric.

In computing U_C, we will evaluate the path integral over the fluctuations of C about the classical solution in a one-loop approximation.[8] The resulting formula

[8]This at least will suffice to show that U_C is nonzero for generic V for all supersymmetric two-

differs only slightly from the formula obtained in section 2 for the superpotential in a model with four unbroken supercharges:

$$U_C = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right)\frac{\mathrm{Pfaff}(\overline{\partial}_{V(-1)})}{(\det{}'\,\overline{\partial}_{\mathcal{O}})^4}\,. \tag{3.2}$$

Only the denominator requires some explanation. Three factors of $\det'\overline{\partial}_{\mathcal{O}}$ arise by interpreting the normal bundle to $\mathbb{R}^6$ as $\mathcal{O}^3$, but the fourth arises in a more complicated way. The normal bundle to C in $Y=$ K3 is as a complex bundle $\mathcal{O}(-2)$, so the bosonic operator for fluctuations of C inside K3 is $\nabla_{\mathcal{O}(-2)} = \partial_{\mathcal{O}(-2)}\overline{\partial}_{\mathcal{O}(-2)}$. Though $\nabla_{\mathcal{O}(-2)}$ has no kernel or cokernel, its left and right-moving factors do. So to factor its determinant, we must use the $\det'$ and write $\det\nabla_{\mathcal{O}(-2)} = \det'\overline{\partial}_{\mathcal{O}(-2)}\det'\partial_{\mathcal{O}(-2)}$. By Serre duality, $\partial_{\mathcal{O}(-2)}$ is the transpose of $\partial_{\mathcal{O}}$, so $\det'\partial_{\mathcal{O}(-2)} = \det'\partial_{\mathcal{O}}$; and likewise $\det'\overline{\partial}_{\mathcal{O}(-2)} = \det'\overline{\partial}_{\mathcal{O}}$. (In the representation of these $\det'$'s as path integrals of $\beta-\gamma$ systems, these statements arise simply from exchanging β and γ.) The factor $\det'\partial_{\mathcal{O}}$ cancels part of the right-moving fermion path integral, and the factor $\det'\overline{\partial}_{\mathcal{O}}$ gives the fourth such factor in the denominator of (3.2).

Much of the discussion of the superpotential in section 2 has a direct analog here. For example, U_C vanishes if and only if $V|_C$ is nontrivial. (In particular, on the $(2,2)$ locus in heterotic string moduli space, where the spin connection is embedded in the gauge group, $V|_C$ is always nontrivial, and hence U_C is identically zero.) Also, while the formula is written for $\mathrm{Spin}(32)/\mathbb{Z}_2$, the analog for $E_8\times E_8$ is determined by arguments similar to those in section 2. Multicovers of C would again be expected to contribute; an analysis of their contributions will be important for applications. Finally, the discussion in section 2.2 is again needed for a precise explanation of the phase of U_C.

One can completely characterize the C's that correct the metric. The group $\Gamma = H_2(Y;\mathbb{Z})$ is a lattice of signature $(3,19)$. A two-sphere $C\subset Y$ that is holomorphic in some complex structure must obey $C\cdot C=-2$. Given a class $x\in\Gamma$ with $x^2=-2$, there is a unique complex structure J on the hyper-Kähler manifold Y (more exactly, a complex structure that is unique up to the possibility of replacing it with the opposite or complex conjugate structure $-J$) for which x is of type (1,1) and so might be the class of a holomorphic curve C, which will automatically have genus zero since $x^2=-2$. In fact, there is a unique two-sphere $C\subset Y$ which, depending on its orientation, has homology class x or $-x$ and is holomorphic with respect to J or $-J$.[9] Hence, the instanton correction to the metric is obtained as a sum over all x with $x^2=-2$.

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spheres C. Additionally, it is quite possible that holomorphy implies vanishing of the higher order corrections. Holomorphy here means really holomorphy on the twistor space of the moduli space. The moduli space itself is a quaternionic manifold, not a complex manifold. The twistor space is obtained by looking at the $(0,4)$ superconformal field theory as a $(0,2)$ model; in other words, a point in the twistor space is a point in the ordinary moduli space together with a choice of a $(0,2)$ subalgebra of the $(0,4)$ superconformal algebra.

[9]This is proved by applying the Riemann-Roch theorem to a holomorphic line bundle $\mathcal{L}$ with

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If the volume of Y in heterotic string units is comparable to $(\alpha')^2$, then many instantons make appreciable contributions to the metric on the moduli space, and the classical formula for this metric will not be a good approximation. Let us ask how, while keeping Y at large volume, the corrections to the metric can become large. This can occur if one of the C's goes to zero volume, which happens precisely when Y develops an A_1 singularity. We also require that the bundle V should be trivial when restricted to C (in the complex structure in which C is holomorphic), and in particular should have vector structure. Under these conditions, the contribution of C and its multicovers to the metric will become large. It would be quite interesting to get a better understanding of this situation.

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$c_1(\mathcal{L}) = x$, getting $h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) = 1$, where we have used the facts that $c_1(Y) = 0$, $c_2(Y) = 24$, and $x^2 = -2$. By Serre duality, $h^2(\mathcal{L}) = h^0(\mathcal{L}^{-1})$. So we get an inequality $h^0(\mathcal{L}) + h^0(\mathcal{L}^{-1}) \geq 1$. A vanishing theorem shows that the sum is precisely 1. So either $\mathcal{L}$ or $\mathcal{L}^{-1}$ has a holomorphic section s, which is unique up to scaling; C is the zero-set of s.

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